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一类线性方程组的特征边界层分析

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摘 要

本文主要研究的是一维线性粘性抛物方程与无粘双曲方程之间的解的渐近极限。我们假定相应的无粘方程的边界是特征的，去研究粘性解与无粘解之间的关系。我们用渐近展开的方法讨论不同区域内粘性方程的近似解，并利用加权能量估计的方法讨论**Prandtl** 型的边界层方程解的存在性，以证明边界层的稳定性。通过对近似解与粘性问题真实解之间的误差进行估计，我们最终得到粘性方程的解与无粘解的渐近等价关系。

关键词

初边值问题，特征边界层，渐近分析，**Prandtl** 型方程，加权能量估计

Analysis of Characteristic Boundary Layers for a Class of Linear Equations

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Abstract

In this paper, we mainly study the asymptotic limit of the solution of the initial boundary value problem for one-dimensional linear equations. We assume that the boundary of the corresponding inviscid equation is characteristic, and study the relationship between the viscous solution and the inviscid one. The boundary layer is characteristic. We use the method of matched asymptotic expansions to discuss the approximate solution of viscous equation in different domains. By using the method of weighted energy estimates, we obtain the existence of solutions for Prandtl type boundary layer equations. In order to prove the stability of the boundary layer, the error between the approximate solution and the real solution of the viscous problem is estimated. Finally, we obtain the asymptotic equivalence between the solutions of the viscous equation and the inviscid one.

Keywords

Initial Boundary Value Problem, Characteristic Boundary Layers, Asymptotic Analysis, Prandtl Type Equations, Weighted Energy Estimate

1. 前言

在本文中, 我们考虑当$\varepsilon \to 0$时, 粘性抛物方程

$$\partial_t u^\varepsilon + A(t,x)\partial_x u^\varepsilon = \varepsilon \partial_x (B(t,x)\partial_x u^\varepsilon), \tag{1.1}$$

$$u^\varepsilon|_{\partial\Omega} = 0, \tag{1.2}$$

$$u^\varepsilon(t=0,x) = u_0^\varepsilon(x), \tag{1.3}$$

解的渐近行为.定义域$\Omega = \{x \in [0,\ +\infty),\ t>0\}$, $\partial\Omega = \{x=0,\ t>0\}$. $A(t,x)$是光滑函数, 且满足

$$A(t,x=0) = 0, \quad \forall t \geq 0, \tag{1.4}$$

由泰勒展式

$$A(t,x)=\varphi(x)\hat{A}(t,x),\quad \forall x, \tag{1.5}$$

$\hat{A}(t,x)$是光滑的, 且

$$\varphi(x)=\frac{x}{1+x}, \tag{1.6}$$

$$A(t,x)|_{\partial\Omega}\neq 0, \tag{1.7}$$

粘性函数$B(t,x)$是光滑函数, 满足一致抛物条件, 即存在$\tilde{c}_0>0$, 使得

$$B(t,x)|\xi|^2\geq \tilde{c}_0|\xi|^2,\quad \forall t,x,u, \tag{1.8}$$

初值u_0^ε满足如下渐近展开

$$\|u_0^\varepsilon(x)-\sum_{i=0}^{m}\sqrt{\varepsilon}^i[u_0^i(x)+u_{b,0}^i(y)]\|_{H^{\tilde{s}}}\leq C\sqrt{\varepsilon}^{m-\tilde{s}},\quad y=\frac{x}{\sqrt{\varepsilon}}, \tag{1.9}$$

其中$u_0^i\in H^\infty(\mathbb{R}_+^1)$, $u_{b,0}^i\in H^\infty(\mathbb{R}_+^1)$, m是足够大的整数, $\tilde{s}$是任意非负整数. 线性抛物型方程组初边值问题的经典结果保证了方程(1.1)-(1.3)在初边值条件满足任意阶相容的前提下, 存在唯一解$u^\varepsilon(t,x)\in H^\infty([0,T_0]\times\mathbb{R}_+^1)$.

一般认为, 粘性抛物方程组的解不能在整个区域上与无粘双曲方程组的解一致接近, 除非这两类方程的边界条件的选择非常特殊. 在许多情况下, 边界条件的差异导致了边界层现象, 这些现象需要用数学理论严格解释并证明, 例如 [1–5]及其参考文献.

根据无粘方程组不同的边界条件, 边界层大致上可以分为两类, 一类是非特征边界, 另一类是特征边界. 在本论文中, 我们将关注方程(1.1)-(1.3)的完全特征边界层的存在性和线性稳定性. 根据假设(1.4), 方程(1.1)-(1.3)对应的无粘方程为

$$\partial_t u^\varepsilon+A(t,x)\partial_x u=0, \tag{1.10}$$

$$u(t=0,x)=u_0^0(x). \tag{1.11}$$

则存在$T_1>0$, 使得方程(1.10)-(1.11)有唯一解$u(t,x)\in H^\infty([0,T_1]\times\mathbb{R}_+^1)$.

我们期望研究抛物方程(1.1)-(1.3)的解u^ε 与无粘方程(1.10)-(1.11)的解u 之间的渐近行为, 时间间隔为$[0,T]$, 且$0<T\leq\min(T_0,T_1)$. 本文安排如下：

首先, 在第二章中我们用多尺度渐近展开 [6]构造粘性方程的近似解, 得到远离边界层的函数及边界层方程. 在特征边界层情况下, 边界层方程不再是常微分方程, 它是一种特殊的退化型偏微分方程, 称为Prandtl 型方程, 见 [7,8]. 第三章讨论了与 [8,9]中类似的一些估计, 这些估计与线性系统有关,其中我们可以得到满足某些衰变特性的边界层函数, 这使得我们可以构造任意阶的粘性方程的近似解.

然后,由于边界是特征的, 粘性系数的ε 阶数不足以抵消特征边界层$\sqrt{\varepsilon}$厚度所带来的奇性, 换言

之, 在非特征情况下, 由于$O(\varepsilon)$粘性项, 通常用Poincaré型不等式控制一阶项是不够的. 由于粘性项中的非线性可能会带来一些困难, 我们在Grenier [9]的启发下对误差项用了加权能量估计, 得到稳定性结果.

2. 近似解的构造

2.1. 多尺度渐近展开

在本节中, 我们利用匹配渐近展开理论构造方程(1.1)的近似解$u_m^\varepsilon(t,x)$. 首先, 我们在远离边界层的区域里作外部展开, 然后在边界层$\partial\Omega=\{x=0,\ t>0\}$ 附近建立边界层展开.

2.1.1. 外部展开

在远离边界层$\partial\Omega=\{x=0,\ t>0\}$的区域里, 方程(1.1)的解可以作如下展开

$$U_m^{IN}(t,x)=\sum_{i=0}^{m}\sqrt{\varepsilon}^i u^i(t,x), \tag{2.1}$$

其中$u^i(t,x), i=0,1,...,m$是确定的. 将近似解(2.1) 代入方程(1.1), 有

$$\begin{aligned}&\frac{\partial u^\varepsilon}{\partial t}+A(t,x)\partial_x u^\varepsilon-\varepsilon\partial_x(B(t,x)\partial_x u^\varepsilon)\\=&(\partial_t u^0+A(t,x)\partial_x u^0)+\sqrt{\varepsilon}(\partial_t u^1+A(t,x)\partial_x u^1)+\sum_{i=2}^{m}\sqrt{\varepsilon}^i(\partial_t u^i+A(t,x)\partial_x u^i+Q^{i-1})+\mathcal{E}_{IN},\end{aligned}$$

其中Q^i仅依赖于$u^k, 0\le k\le i-1$, 余项$\mathcal{E}_{IN}$ 满足

$$|\mathcal{E}_{IN}|_{L^\infty(\mathbb{R}_+^1)}\le C\sqrt{\varepsilon}^{m+1}, \tag{2.2}$$

且

$$\|\partial_x^\alpha\partial_t^\beta\mathcal{E}_{IN}\|_{L^2(\mathbb{R}_+^1)}^2\le C\varepsilon^{m+1},\quad \forall\alpha\ge 0,\quad \forall\beta\ge 0. \tag{2.3}$$

根据系数$\sqrt{\varepsilon}$ 的幂次进行分类, 得

$$O(1):\partial_t u^0+A(t,x)\partial_x u^0=0, \tag{2.4}$$
$$O(\sqrt{\varepsilon}):\partial_t u^1+A(t,x)\partial_x u^1=0, \tag{2.5}$$
$$O(\sqrt{\varepsilon}^i):\partial_t u^i+A(t,x)\partial_x u^i+Q^{i-1}=0, \tag{2.6}$$

其中$i=2,...,m$. 同时得到初值条件

$$u^0(t=0,x)=u_0^0(x), \tag{2.7}$$
$$u^1(t=0,x)=u_0^1(x), \tag{2.8}$$
$$u^i(t=0,x)=u_0^i(x),\quad i=2,...,m. \tag{2.9}$$

我们发现方程(2.4)的解u^0是无粘方程(1.10)-(1.11) 的解, 因此令$u^0(t,x)$ 是无粘方程(1.10)-(1.11)的光滑解$u\in H^\infty([0,T]\times\mathbb{R}_+^1)$. 由假设(1.4), 方程(2.5)-(2.9) 存在唯一的解$u^i\in H^\infty([0,T]\times\mathbb{R}_+^1)$.

2.1.2. 边界层展开

在边界层$\partial\Omega = \{x = 0,\ t > 0\}$ 附近, 我们设近似解为

$$u_m^\varepsilon(t,x) = U_m^{IN}(t,x) + U_m^B(t,y) = \sum_{i=0}^{m} \sqrt{\varepsilon}^i (u^i(t,x) + u_b^i(t,y)), \tag{2.10}$$

其中$y = \frac{x}{\sqrt{\varepsilon}}$. 将$u_m^\varepsilon$代入方程(1.1), 有

$$\begin{aligned}&\frac{\partial u^\varepsilon}{\partial t} + A(t,x)\partial_x u^\varepsilon - \varepsilon\partial_x(B(t,x)\partial_x u^\varepsilon)\\ =&\partial_t u_b^0 + y\hat{A}(t,0)\partial_y u_b^0 - \partial_y(B(t,0)\partial_y u_b^0)\\ &+\sum_{i=1}^{m}\sqrt{\varepsilon}^i(\partial_t u_b^i + y\hat{A}(t,0)\partial_y u_b^i - \partial_y(B(t,0)\partial_y u_b^i + \tilde{Q}^{i-1}) - \mathcal{E}_B - \mathcal{E}_{IN}\end{aligned}$$

其中余项$\mathcal{E}_B$满足

$$|\mathcal{E}_B|_{L^\infty(\mathbb{R}_+^1)} \le C\sqrt{\varepsilon}^{m+1}, \tag{2.11}$$

且

$$\|x^\alpha \partial_x^\alpha \partial_t^\beta \mathcal{E}_B\|_{L^2(\mathbb{R}_+^1)}^2 \le C\varepsilon^{m+\frac{3}{2}}, \quad \alpha \ge 0, \forall\beta \ge 0. \tag{2.12}$$

按照系数$\sqrt{\varepsilon}$ 的幂次分类, 得

$$O(1): \partial_t u_b^0 + y\hat{A}(t,0)\partial_y u_b^0 - \partial_y(B(t,0)\partial_y u_b^0) = 0, \tag{2.13}$$

$$O(\sqrt{\varepsilon}^i): \partial_t u_b^i + y\hat{A}(t,0)\partial_y u_b^i - \partial_y(B(t,0)\partial_y u_b^i) + \tilde{Q}^{i-1} = 0 \tag{2.14}$$

其中$i = 1,2,...,m$, $\tilde{Q}^i$依赖于$u^k, u_b^k, 0 \le k \le i-1$. 同时得到初边值条件

$$u_b^0(t=0,y) = u_{b,0}^0(y), \quad u_b^0(t,0) = u^0(t,0), \quad u_b^0(t, y\to+\infty) = 0, \tag{2.15}$$

$$u_b^i(t=0,y) = u_{b,0}^i(y), \quad u_b^i(t,0) = u^i(t,0), \quad u_b^i(t, y\to+\infty) = 0, \tag{2.16}$$

成立,$1 \le i \le m$.

2.2. Prandtl型边界层方程

2.2.1. 线性边界层方程

对于Prandtl 型边界层方程初边值问题, 将利用加权能量估计来讨论解的存在性.

$$\partial_t w + y\hat{A}(t,0)\partial_y w - \partial_y(B(t,0)\partial_y w) = 0, \tag{2.17}$$

$$w(t, y=0) = \bar{w}(t), \tag{2.18}$$

$$w(t, y\to+\infty) = 0, \tag{2.19}$$

$$w(t=0,y) = w_0(y). \tag{2.20}$$

其中$y = \frac{x}{\sqrt{\varepsilon}}$, $\bar{w}(t) \in H^s([0,T])$, s 足够大. 为了证明初边值问题(2.17)-(2.20) 解的存在性, 我们必须处理方程中的无界项$y\hat{A}\partial_y w$. 由于$y = 0$ 时, 方程(2.17)是退化的, 并且当$y\to\infty$时,$y\hat{A}\partial_y w$ 是无界的, 这使得普通的Sobolev 范数不足以去控制方程的解. 为此，我们对非负整数α 和β 引入加权范数

$$|||w|||_s^2 = \sum_{\substack{\alpha\in\mathcal{N},\beta\in\mathcal{N}\\ 0\le\alpha+\beta\le s}} C_0^{-\alpha-\beta}|||w|||_{\alpha,\beta}^2, \tag{2.21}$$

其中

$$|||w|||_{\alpha,\beta}^2 = \int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} y^{2\alpha}|\partial_y^\alpha\partial_t^\beta w|^2 dydt, \tag{2.22}$$

C_0是一个常数. 为了方程在分部积分时没有边界项, 我们引入截断函数, 设$\eta(y)\in C^\infty(\mathbb{R}_+^1)$ 是截断函数, 且满足

$$\eta(y_1) = \begin{cases} 1, & 0\le y\le 1, \\ \text{单调递减}, & 1<y<2, \end{cases} \tag{2.23}$$

令

$$z(t,y) = w(t,y) - \tilde{w}(t,y), \tag{2.24}$$

其中$\tilde{w}(t,y)=\eta(y)\bar{w}(t)$, 则$z(t,y)$ 满足

$$\partial_t z + y\hat{A}(t,0)\partial_y z - \partial_y(B(t,0)\partial_y z) = \sigma, \tag{2.25}$$

$$z(t,y=0) = 0, \tag{2.26}$$

$$z(t,y\to+\infty) = 0, \tag{2.27}$$

$$z(t=0,y) = w_0(y) - \eta(y)\bar{w}(0), \tag{2.28}$$

这里

$$\sigma = -\eta(y)\partial_t\bar{w} - y\hat{A}(t,0)\eta'(y)\bar{w} + \partial_y(B(t,0)\eta'(y)\bar{w}).$$

则对任意的$\alpha,\beta\le s-1$, 有

$$\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} y^{2\alpha}|\partial_y^\alpha\partial_t^\beta \sigma|^2 dydt \le C||\bar{w}||_{H^s}, \tag{2.29}$$

定理2.1: 方程(2.25)-(2.28)存在唯一解$z\in H^s([0,T]\times\mathbb{R}_+^1)$, 使得

$$|||z|||_s^2 \le C, \tag{2.30}$$

这里C依赖于$||\bar{w}||_{H^s}$.

证明: 关于方程(2.25)两边同时作用算子$\partial_y^\alpha\partial_t^\beta$, 得

$$\partial_t|||z|||_{\alpha,\beta}^2 = 2\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} y^{2\alpha}\partial_y^\alpha\partial_t^\beta z\cdot\partial_y^\alpha\partial_t^{\beta+1}z dydt = 2\sum_{i=1}^5 I_i, \tag{2.31}$$

其中

$$I_1 = -\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} y^{2\alpha}\partial_y^\alpha\partial_t^\beta z\cdot y\hat{A}(t,0)\partial_y^{\alpha+1}\partial_t^\beta z dydt,$$

$$I_2 = -\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} y^{2\alpha}\partial_y^\alpha\partial_t^\beta z\cdot[y\hat{A}\partial_y, \partial_y^\alpha\partial_t^\beta]z dydt,$$

$$I_3 = \int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} y^{2\alpha}\partial_y^\alpha\partial_t^\beta z\cdot B\partial_y^{\alpha+2}\partial_t^\beta z dydt,$$

$$I_4 = \int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} y^{2\alpha}\partial_y^\alpha\partial_t^\beta z\cdot[\partial_y(B\partial_y), \partial_y^\alpha\partial_t^\beta]z dydt,$$

$$I_5 = \int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} y^{2\alpha}\partial_y^\alpha\partial_t^\beta z\cdot\partial_y^\alpha\partial_t^\beta\sigma dydt.$$

对变量y用分部积分, 得

$$
\begin{aligned}
I_1 =& \int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} \partial_y^\alpha \partial_t^\beta z \cdot \partial_y(\hat{A} y^{2\alpha+1} \partial_y^\alpha \partial_t^\beta z) dydt \\
=& \frac{1}{2}(2\alpha+1) \int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} \partial_y^\alpha \partial_t^\beta z \cdot y_1^{2\alpha} \hat{A} \partial_y^\alpha \partial_t^\beta z dydt,
\end{aligned}
$$

由$|\hat{A}(t,0)| \le C$, 得

$$
|I_1| \le C|||z|||_{\alpha,\beta}^2. \tag{2.32}
$$

交换子I_2是如下各项的总和

$$
J = \int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} y^{2\alpha} \partial_y^\alpha \partial_t^\beta z \cdot \partial_y^{\alpha'} \partial_t^{\beta'}(y\hat{A}) \cdot \partial_y^{\alpha-\alpha'+1} \partial_t^{\beta-\beta'} z dydt,
$$

其中$\alpha' + \beta' \ge 1$, $0 \le \alpha' \le \alpha$, $0 \le \beta' \le \beta$. 若$\alpha' = 0$, 则$\beta \ge \beta' \ge 1$. 于是, 有

$$
\begin{aligned}
J =& \int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} y^{2\alpha} \partial_y^\alpha \partial_t^\beta z \cdot y \partial_t^{\beta'} \hat{A} \cdot \partial_y^{\alpha+1} \partial_t^{\beta-\beta'} z dydt \\
=& \int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} y^{\alpha} \partial_y^\alpha \partial_t^\beta z \cdot \partial_t^{\beta'} \hat{A} \cdot y^{\alpha+1} \partial_y^{\alpha+1} \partial_t^{\beta-\beta'} z dydt,
\end{aligned}
$$

因为$|\partial_t^{\beta'} \hat{A}| \le C$, 所以

$$
|J| \le C|||z|||_{\alpha,\beta}^2 + C|||z|||_{\alpha+1,\beta-\beta'}^2 \le C|||z|||_{\alpha+\beta}^2, \tag{2.33}
$$

若$\alpha' = 1$, 则

$$
J = \int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} y^{2\alpha} \partial_y^\alpha \partial_t^\beta z \cdot \partial_t^{\beta'} \hat{A} \cdot \partial_y^{\alpha} \partial_t^{\beta-\beta'} z dydt,
$$

由$|\partial_t^{\beta'} \hat{A}| \le C$, 且利用Cauchy 不等式, 可得

$$
|J| \le C|||z|||_{\alpha,\beta}^2 + C|||z|||_{\alpha,\beta-\beta'}^2 \le C|||z|||_{\alpha+\beta}^2, \tag{2.34}
$$

若$\alpha' > 1$, 则$J = 0$. 综合(2.33)和(2.34),

$$
|I_2| \le \sum |J| \le C|||z|||_{\alpha+\beta}^2. \tag{2.35}
$$

对变量y用分部积分,得

$$
\begin{aligned}
I_3 =& -2\alpha \int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} y^{2\alpha-1} \partial_y^\alpha \partial_t^\beta z \cdot B \partial_y^{\alpha+1} \partial_t^\beta z dydt \\
& - \int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} y^{2\alpha} \partial_{y_1}^{\alpha+1} \partial_t^\beta z \cdot B \partial_y^{\alpha+1} \partial_t^\beta z dydt,
\end{aligned}
$$

若$\alpha = 0$, 则

$$
I_3 = -\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} \partial_y \partial_t^\beta z \cdot B \cdot \partial_y \partial_t^\beta z dydt,
$$

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$$|I_3| \le C\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} |\partial_y\partial_t^\beta z|^2 dydt, \tag{2.36}$$

若$\alpha\ge 1$,

$$\begin{aligned}&|I_3| + \tilde{c}_0\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} y^{2\alpha}|\partial_y^{\alpha+1}\partial_t^\beta z|^2 dydt\\ \le& 2\alpha|\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} y^{2\alpha-1}\partial_y^\alpha\partial_t^\beta z\cdot B\partial_y^{\alpha+1}\partial_t^\beta z dydt|\\ \le& C\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} |y^{\alpha-1}\partial_y^\alpha\partial_t^\beta z\cdot y^\alpha\partial_y^{\alpha+1}\partial_t^\beta z|dydt,\end{aligned}$$

由Cauchy不等式得

$$|I_3| \le C\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} y^{2(\alpha-1)}|\partial_y^\alpha\partial_t^\beta z|^2 dydt + \tau\tilde{c}_0\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} y^{2\alpha}|\partial_y^{\alpha+1}\partial_t^\beta z|^2 dydt,$$

取τ足够小,

$$|I_3| + \tilde{c}_0\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} y^{2\alpha}|\partial_y^{\alpha+1}\partial_t^\beta z|^2 dydt \le C\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} y^{2(\alpha-1)}|\partial_y^\alpha\partial_t^\beta z|^2 dydt, \tag{2.37}$$

综合(2.36)和(2.37), 有

$$\begin{aligned}&|I_3| + \tilde{c}_0\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} y^{2\alpha}|\partial_y^{\alpha+1}\partial_t^\beta z|^2 dydt\\ &\le C\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} y^{2(\alpha-1)}|\partial_y^\alpha\partial_t^\beta z|^2 dydt + C\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} |\partial_y\partial_t^\beta z|^2 dydt,\end{aligned} \tag{2.38}$$

交换子I_4是如下各项的总和

$$J = \int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} y^{2\alpha}\partial_y^\alpha\partial_t^\beta z\cdot\partial_y^{\alpha'}\partial_t^{\beta'}B\cdot\partial_y^{\alpha-\alpha'+2}\partial_t^{\beta-\beta'} z dydt,$$

其中$\alpha'+\beta'\ge 1$. 若$\alpha=0$, 则$\alpha'=0$, $\beta'\ge 1$,

$$J = \int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} \partial_t^\beta z\cdot\partial_t^{\beta'}B\cdot\partial_y^2\partial_t^{\beta-\beta'} z dydt,$$

利用分部积分, 得

$$J = \int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} \partial_y\partial_t^\beta z\cdot\partial_t^{\beta'}B\cdot\partial_y\partial_t^{\beta-\beta'} z dydt,$$

由$|\partial_t^{\beta'}B|\le C$ 与Cauchy 不等式, 得

$$\begin{aligned}|J| \le& C\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} |\partial_y\partial_t^\beta z\cdot\partial_y\partial_t^{\beta-\beta'} z|dydt\\ \le& C\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} |\partial_y\partial_t^\beta z|^2 dydt + C\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} |\partial_y\partial_t^{\beta-\beta'} z|^2 dydt,\end{aligned} \tag{2.39}$$

若$\alpha\ge 1$, $\partial_y^{\alpha'}\partial_t^{\beta'}B(t,0)=0$, 则$J=0$. 故

$$|I_4| \le C\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} |\partial_y\partial_t^\beta z|^2 dydt + C\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} |\partial_y\partial_t^{\beta-\beta'} z|^2 dydt, \tag{2.40}$$

利用Cauchy不等式

$$|I_5| \le C|||z|||_{\alpha,\beta}^2 + C|||\sigma|||_{\alpha,\beta}^2 \le C|||z|||_{\alpha,\beta}^2 + C||\bar{w}||_{H^s}. \tag{2.41}$$

综合(2.32), (2.35), (2.38), (2.40) 以及(2.41)式, 得到

$$
\begin{aligned}
&\partial_t|||z|||^2_{\alpha,\beta}+\tilde{c}_0\int_{\mathbb{R}^1_+}\int_{\mathbb{R}^1_+}y^{2\alpha}|\partial_y^{\alpha+1}\partial_t^{\beta}z|^2dydt\\
\leq& C|||z|||^2_{\alpha,\beta}+C|||z|||^2_{\alpha+\beta}+C||\bar{w}||_{H^s}\\
&+C\int_{\mathbb{R}^1_+}\int_{\mathbb{R}^1_+}y^{2(\alpha-1)}|\partial_y^{\alpha}\partial_t^{\beta}z|^2dydt+C\int_{\mathbb{R}^1_+}\int_{\mathbb{R}^1_+}|\partial_y\partial_t^{\beta-\beta'}z|^2dydt,
\end{aligned}\tag{2.42}
$$

由加权范数的定义, 我们对(2.42)式两边同乘以$C_0^{-\alpha-\beta}$再求和, 选择足够大的C_0, 使得(2.43) 等式右边的最后两项可以被左边第二项控制. 于是有

$$
\partial_t|||z|||^2_s\leq C|||z|||^2_s+C||\bar{w}||_{H^s}\leq C(|||z|||^2_s+1),\tag{2.43}
$$

由Gronwall不等式, 得

$$
|||z|||^2_s\leq C.
$$

定理得证.

命题2.1: 设s充分大, 则

$$
|\partial_y^{\alpha}\partial_t^{\beta}w|_{L^\infty(\Omega)}\leq C|||w|||_s,\tag{2.44}
$$

其中$\alpha+\beta\leq\frac{s+1}{2}$.

证明: 由定理2.1知$|||z|||_s$ 在$[0,T]$ 有界. 又由于(2.24)得$|||w|||_s$ 在$[0,T]$有界. 将方程(2.13)变形为

$$
\partial_y^2w=M\partial_tw+yN\partial_yw,\tag{2.45}
$$

其中$M=B^{-1}(t,0)$, $N=B^{-1}(t,0)\hat{A}(t,0)$, 关于$M,N$, 我们作如下假设

$$
|\partial_t^{\beta}M|_{L^\infty(\Omega)}\leq C_1,\tag{2.46}
$$

$$
|\partial_t^{\beta}N|_{L^\infty(\Omega)}\leq C_2.\tag{2.47}
$$

对方程(2.45)两边同时作用算子$y^\alpha\partial_y^\alpha\partial_t^\beta$

$$
y^\alpha\partial_y^\alpha\partial_t^\beta\partial_y^2w=y^\alpha\partial_y^\alpha\partial_t^\beta(M\partial_tw)+y^\alpha\partial_y^\alpha\partial_t^\beta(yN\partial_yw)\tag{2.48}
$$

我们希望$||y^\alpha\partial_y^\alpha\partial_t^\beta\partial_y^2w||_{L^2}$ 有界. 因此观察(2.48)式右边每一项是否有界. 其中第一项为

$$
\begin{aligned}
&\int_{\mathbb{R}^1_+}\int_{\mathbb{R}^1_+}|y^\alpha\partial_y^\alpha\partial_t^\beta(M\partial_tw)|^2dydt\\
=&\int_{\mathbb{R}^1_+}\int_{\mathbb{R}^1_+}|My^\alpha\partial_y^\alpha\partial_t^{\beta+1}w|^2dydt+\int_{\mathbb{R}^1_+}\int_{\mathbb{R}^1_+}|[M\partial_t,y^\alpha\partial_y^\alpha\partial_t^\beta]w|^2dydt\\
\triangleq&E_1+E_2,
\end{aligned}
$$

显然, $|E_1|\leq C|||w|||^2_s$. 交换子E_2是如下各项的和

$$
J=\int_{\mathbb{R}^1_+}\int_{\mathbb{R}^1_+}|y^\alpha\partial_y^{\alpha'}\partial_t^{\beta'}M\cdot\partial_y^{\alpha-\alpha'}\partial_t^{\beta-\beta'+1}w|^2dydt,
$$

其中$\alpha'+\beta'\geq1$. 当$\alpha'=0$ 时,

$$
\begin{aligned}
J=&\int_{\mathbb{R}^1_+}\int_{\mathbb{R}^1_+}|y^\alpha\partial_t^{\beta'}M\cdot\partial_y^{\alpha}\partial_t^{\beta-\beta'+1}w|^2dydt\\
\leq& C|||w|||^2_{\alpha,\beta-\beta'+1}\leq C|||w|||^2_s,
\end{aligned}\tag{2.49}
$$

其中$0 \le \alpha + \beta - \beta' \le s-1$. 当$\alpha' = 1$ 时, $J=0$. 由(2.49)可得

$$E_2 \le C|||w|||_s^2,$$

因此,

$$\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} |y^\alpha \partial_y^\alpha \partial_t^\beta (M\partial_t w)|^2 dydt \le C|||w|||_s^2,$$

同理,

$$\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} |y^\alpha \partial_y^\alpha \partial_t^\beta (yN\partial_y w)|^2 dydt \le C|||w|||_s^2,$$

这样就有

$$||y^\alpha \partial_y^{\alpha+2} \partial_t^\beta w||_{L^2}^2 \le C|||w|||_s^2, \tag{2.50}$$

其中$\alpha+\beta \le s-1$. 同样地, 我们对方程(2.45)两边同时作用算子$y^\alpha \partial_y^{\alpha+1}\partial_t^\beta$, 可得

$$||y^\alpha \partial_y^{\alpha+3} \partial_t^\beta w||_{L^2}^2 \le C|||w|||_s^2, \tag{2.51}$$

其中$\alpha+\beta \le s-2$, 依此类推,

$$||y^\alpha \partial_y^{\alpha+k} \partial_t^\beta w||_{L^2}^2 \le C|||w|||_s^2, \tag{2.52}$$

其中$\alpha+\beta \le s-k+1$. 令$\alpha = 0$,

$$||\partial_y^k \partial_t^\beta w||_{L^2}^2 \le C|||w|||_s^2,$$

得到$w \in H^m = H^{s-k+1}$. 让s充分大, 有

$$|\partial_y^\alpha \partial_t^\beta w|_{L^\infty(\Omega)} \le C|||w|||_s.$$

这样命题2.1得证. 因此在Sobolev 空间中, 我们得到了关于线性边界层方程初边值问题解的存在性和稳定性.

2.3. 误差方程

由定理2.1: 当s充分大, 我们得到了Prandtl 型边界层方程初边值问题的光滑解$u_b^0(t,y) \in H^\infty([0,T]\times\mathbb{R}_+^1)$, $u_b^i(t,y) \in H^\infty([0,T]\times\mathbb{R}_+^1), 1 \le i \le m$. 因此, 近似解$u_m^\varepsilon$满足

$$\partial_t u_m^\varepsilon + A(t,x)\partial_x u_m^\varepsilon - \varepsilon\partial_x(B(t,x)\partial_x u_m^\varepsilon) = R^\varepsilon, \tag{2.53}$$

$$u_m^\varepsilon|_{\partial\Omega} = 0, \tag{2.54}$$

$$u_m^\varepsilon(t=0,x) = u_0^\varepsilon(x), \tag{2.55}$$

其中$R^\varepsilon = \mathcal{E}_{IN} + \mathcal{E}_B$, $\mathcal{E}_{IN}$ 和$\mathcal{E}_B$是内展开和边界层展开的余项. 因此, R^ε 满足

$$||\varphi^\alpha(x)\partial_x^\alpha \partial_t^\beta R^\varepsilon||_{L^2(\Omega)}^2 \le C\varepsilon^{m+\frac{3}{2}}, \forall \alpha+\beta \le s. \tag{2.56}$$

令$v = u^\varepsilon - u_m^\varepsilon$, 则

$$\frac{\partial v}{\partial t} + A(t,x)\partial_x v - \varepsilon\partial_x(B(t,x)\partial_x v) = -R^\varepsilon, \tag{2.57}$$

$$v|_{\partial\Omega} = 0, \tag{2.58}$$

$$v(t=0,x) = 0. \tag{2.59}$$

在下一个章节中, 我们将对线性误差方程进行稳定性分析.

3. 稳定性分析

下面证明线性方程(1.1)-(1.3)的稳定性,

$$\partial_t v + A(t,x)\partial_x v - \varepsilon\partial_x(B(t,x)\partial_x v) = -R^\varepsilon(t,x), \tag{3.1}$$

$$v|_{\partial\Omega} = 0, \tag{3.2}$$

$$v(t=0,x) = 0, \tag{3.3}$$

其中$\Omega = \{x \in [0,\ +\infty),\ t > 0\}$, $\partial\Omega = \{x = 0,\ t > 0\}$. 为了估计误差, 我们引入加权范数

$$|||v|||_s^2 = \sum_{\substack{\alpha\in\mathcal{N},\beta\in\mathcal{N}\\0\le\alpha+\beta\le s}} C_0^{-\alpha-\beta}|||v|||_{\alpha,\beta}^2, \tag{3.4}$$

$$|||v|||_{\alpha,\beta}^2 = ||\varphi^\alpha(x)\partial_x^\alpha\partial_t^\beta v||_{L^2(\Omega)}^2 \tag{3.5}$$

其中$\varphi(x) = \frac{x}{1+x}$.

定理3.1: 设任意的$0 \le \alpha+\beta \le s$,

$$||\partial_t^\beta A(t,x)||_{L^\infty(\Omega)} \le C\varphi(x), \tag{3.6}$$

$$||\partial_x^\alpha\partial_t^\beta A(t,x)||_{L^\infty(\Omega)} \le C + C|\sqrt{\varepsilon}|^{-\alpha+1}\theta(\frac{x}{\sqrt{\varepsilon}}), \tag{3.7}$$

$$||\partial_x^\alpha\partial_t^\beta B(t,x)||_{L^\infty(\Omega)} \le C + C|\sqrt{\varepsilon}|^{-\alpha}\theta(\frac{x}{\sqrt{\varepsilon}}), \tag{3.8}$$

$\theta(x) \ge 0$是光滑函数, 且对任意的$x \in \mathbb{R}_+^1$和n, $|x^n\theta(x)| \le C_n$, 则方程组(3.1)-(3.3) 有唯一解$v(x,t) \in H^s(\Omega)$, 使得

$$|||v|||_s^2 \le C\varepsilon^{m+\frac{3}{2}}. \tag{3.9}$$

证明:

由计算可得

$$\partial_t|||v|||_{\alpha,\beta}^2 = 2\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1}\varphi^{2\alpha}\partial_x^\alpha\partial_t^\beta\cdot\partial_x^\alpha\partial_t^{\beta+1}vdxdt = 2\sum_{i=1}^5 I_i$$

其中

$$I_1 = -\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1}\varphi^{2\alpha}\partial_x^\alpha\partial_t^\beta v\cdot A\partial_x^{\alpha+1}\partial_t^\beta vdxdt,$$

$$I_2 = -\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1}\varphi^{2\alpha}\partial_x^\alpha\partial_t^\beta v\cdot[A\partial_x,\partial_x^\alpha\partial_t^\beta]vdxdt,$$

$$I_3 = \varepsilon\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1}\varphi^{2\alpha}\partial_x^\alpha\partial_t^\beta v\cdot B\partial_x^{\alpha+2}\partial_t^\beta vdxdt,$$

$$I_4 = \varepsilon\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1}\varphi^{2\alpha}\partial_x^\alpha\partial_t^\beta v\cdot[\partial_x(B\partial_x),\partial_x^\alpha\partial_t^\beta]vdxdt,$$

$$I_5 = -\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1}\varphi^{2\alpha}\partial_x^\alpha\partial_t^\beta v\cdot\partial_x^\alpha\partial_t^\beta R^\varepsilon dxdt,$$

由分部积分得

$$I_1 = \frac{1}{2}\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} \partial_x^\alpha \partial_t^\beta v \cdot \partial_x(\varphi^{2\alpha}A)\partial_x^\alpha\partial_t^\beta v dxdt,$$

由(3.7)得$||\partial_x A(t,x)|| \le C$, 从而$||\varphi^{2\alpha}\partial_x A|| \le C\varphi^{2\alpha}$. 又$|A(t,x)| \le Cx$, 且$|x\partial_x\varphi^{2\alpha}| \le C\varphi^{2\alpha}$, 所以

$$||\partial_x(\varphi^{2\alpha}A)|| \le C\varphi^{2\alpha},$$

故

$$|I_1| \le C|||v|||_{\alpha,\beta}^2. \tag{3.10}$$

交换子I_2是下面各项的总和

$$J = \int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} \varphi^{2\alpha}\partial_x^\alpha\partial_t^\beta v \cdot \partial_x^{\alpha'}\partial_t^{\beta'} A \cdot \partial_x^{\alpha-\alpha'+1}\partial_t^{\beta-\beta'} v dxdt,$$

其中$\alpha' + \beta' \ge 1$, $0 \le \alpha' \le \alpha$, $0 \le \beta' \le \beta$.

若$\alpha' = 0$, 由(3.6)

$$\begin{aligned}|J| &\le C\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} |\varphi^\alpha\partial_x^\alpha\partial_t^\beta v||\varphi^{\alpha+1}\partial_x^{\alpha+1}\partial_t^{\beta-\beta'} v|dxdt \\ &\le C|||v|||_{\alpha,\beta}^2 + C|||v|||_{\alpha+1,\beta-\beta'}^2 \\ &\le C|||v|||_{\alpha+\beta}^2,\end{aligned} \tag{3.11}$$

若$\alpha' \ge 1$, 由(3.7)得$\varphi^{2\alpha}||\partial_x^{\alpha'}\partial_t^{\beta'} A|| \le C\varphi^{2\alpha} \le C\varphi^{2\alpha-\alpha'+1}$,

$$\begin{aligned}|J| &\le C\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} |\varphi^\alpha\partial_x^\alpha\partial_t^\beta v||\varphi^{\alpha-\alpha'+1}\partial_x^{\alpha-\alpha'+1}\partial_t^{\beta-\beta'} v|dxdt \\ &\le C|||v|||_{\alpha,\beta}^2 + C|||v|||_{\alpha-\alpha'+1,\beta-\beta'}^2 \\ &\le C|||v|||_{\alpha+\beta}^2,\end{aligned} \tag{3.12}$$

综合(3.11), (3.12)得

$$|I_2| \le C|||v|||_{\alpha+\beta}^2. \tag{3.13}$$

对x利用分部积分可得

$$\begin{aligned}I_3 = &-\varepsilon\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} \partial_x^\alpha\partial_t^\beta v \cdot \partial_x(\varphi^{2\alpha}B) \cdot \partial_x^{\alpha+1}\partial_t^\beta v dxdt \\ &-\varepsilon\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} B\partial_x^{\alpha+1}\partial_t^\beta v \cdot \varphi^{2\alpha} \cdot \partial_x^{\alpha+1}\partial_t^\beta v dxdt,\end{aligned} \tag{3.14}$$

令

$$\xi = \nabla(\partial_x^\alpha\partial_t^\beta v)$$

由一致抛物条件, 结合(3.14)得到

$$\begin{aligned}&|I_3| + \widetilde{c}_0\varepsilon\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} |\xi|^2 dxdt \\ \le &-\varepsilon\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1} \partial_x^\alpha\partial_t^\beta v \cdot \partial_x(\varphi^{2\alpha}B) \cdot \partial_x^{\alpha+1}\partial_t^\beta v dxdt\end{aligned}$$

若$\alpha=0, \beta\le s-1$, 并由Cauchy 不等式得

$$|I_3|+\widetilde{c}_0\varepsilon\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1}|\xi|^2dxdt\le\tau C\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1}|\partial_x\partial_t^\beta v|^2dxdt$$

若$\alpha_1\ge 1$, 由Cauchy 不等式得

$$\begin{aligned}&\varepsilon\Big|\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1}\partial_x\partial_t^\beta v\cdot\partial_x(\varphi^{2\alpha}B)\cdot\partial_x^{\alpha+1}\partial_t^\beta vdxdt\Big|\\&\le C\varepsilon\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1}\varphi^{2\alpha-2}|\partial_x\partial_t^\beta v|^2dxdt+C|||v|||^2_{\alpha,\beta}\\&+\tau\widetilde{c}_0\varepsilon\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1}\varphi^{2\alpha}|\partial_x^{\alpha+1}\partial_t^\beta v|^2dxdt,\end{aligned}\tag{3.15}$$

取τ 足够小, 由(3.15), 得

$$|I_3|+\varepsilon\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1}\varphi^{2\alpha}|\xi|^2dxdt\le C\varepsilon\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1}\varphi^{2(\alpha-1)}|\partial_t^\beta v|^2dxdt+C|||v|||^2_{\alpha,\beta}.\tag{3.16}$$

交换子I_4是如下各项的总和

$$J=\varepsilon\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1}\varphi^{2\alpha}\partial_x\partial_t^\beta v\cdot\partial_x^{\alpha'+1}\partial_t^{\beta'}B\cdot\partial_x^{\alpha-\alpha'+1}\partial_t^{\beta-\beta'}vdxdt,$$

其中$\alpha'+\beta'\ge 1$, $0\le\alpha'\le\alpha$, $0\le\beta'\le\beta$.

若$\alpha'=0$, 由(3.8)

$$|J|\le+C\varepsilon\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1}|\partial_x\partial_t^{\beta-\beta'}v|^2dxdt,\tag{3.17}$$

若$\alpha_1'\ge 1$, 同理

$$\begin{aligned}|J|\le&C\varepsilon\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1}\varphi^{2\alpha-\alpha'}|\partial_x\partial_t^\beta v\cdot\partial_x^{\alpha-\alpha'+1}\partial_t^{\beta-\beta'}v|dxdt\\\le&C\varepsilon\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1}\varphi^{2(\alpha-\alpha'+1)}|\partial_x^{\alpha-\alpha'+1}\partial_t^{\beta-\beta'}v|^2dxdt\\&+C\varepsilon\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1}\varphi^{2(\alpha-1)}|\partial_x\partial_t^\beta v|^2dxdt\end{aligned}\tag{3.18}$$

故, 综合(3.17), (3.18)

$$\begin{aligned}|I_4|\le&C\varepsilon\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1}\varphi^{2(\alpha-1)}|\partial_x\partial_t^\beta v|^2dxdt+C\varepsilon\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1}|\partial_x\partial_t^{\beta-\beta'}v|^2dxdt\\&+C\varepsilon\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1}|\partial_t^\beta v|^2dxdt+C|||v|||^2_{\alpha+\beta}.\end{aligned}\tag{3.19}$$

由(2.56)得

$$|I_5|\le C|||v|||^2_{\alpha,\beta}+C\varepsilon^{m+\frac{3}{2}}.\tag{3.20}$$

利用$I_i, 1\le i\le 5$ 的所有估计

$$\begin{aligned}&\partial_t|||v|||^2_{\alpha,\beta}+\varepsilon\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1}\varphi^{2\alpha}|\xi|^2dxdt\\\le&C|||v|||^2_{\alpha,\beta}+C|||v|||^2_{\alpha+\beta}+C\varepsilon^{m+\frac{3}{2}}\\&+C\varepsilon\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1}\varphi^{2(\alpha-1)}|\partial_x\partial_t^\beta v|^2dxdt+C\varepsilon\int_{\mathbb{R}_+^1}\int_{\mathbb{R}_+^1}|\partial_x\partial_t^{\beta-\beta'}v|^2dxdt,\end{aligned}$$

根据加权范数的定义, 我们对上式两边同乘以$C_0^{-\alpha-\beta}$, 并求和, 并选择足够大的C_0, 使得上式右边最后两项可以被左边第二项控制. 因此, 我们有

$$\partial_t |||v|||_s^2 \leq C|||v|||_s^2 + C\varepsilon^{m+\frac{3}{2}}, \tag{3.21}$$

由Gronwall不等式得

$$|||v|||_s^2 \leq C. \tag{3.22}$$

4. 结论

在本文中，首先, 我们用多尺度渐近展开构造粘性方程的近似解, 得到远离边界层的函数及边界层方程. 其次讨论了一些估计, 这使得我们可以构造任意阶的粘性方程的近似解.最后,由于边界是特征的, 粘性系数的ε 阶数不足以抵消特征边界层$\sqrt{\varepsilon}$ 厚度所带来的奇性, 我们对误差项用了加权能量估计,得到了同一粘性系统的近似解. 这最终证明在远离边界的区域内, 粘性方程的解收敛到无粘方程的解是L^∞收敛, 在边界层附近, u^ε 有一个急剧的变化, 只能做到L^2 收敛.

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